

Introduction

Basic Idea



Either you are a natural born lightning calculator (LEIBNIZ, EULER) or you know many fast algorithms to choose problem-specifically the best one, instead of having only few general algorithms.

- ▶ this talk is mainly limited to numbers with two digits
- ▶ derivations of the formulas is simple and left as exercises for the interested audience

Notation:

double bar divides two digits $a_3a_2||a_1a_0 = a_3a_2a_1a_0$,
single bar divides one digit $a_1|a_0 = a_1a_0$.

Preliminaries

Mind vs. Machine



Assumptions:

- ▶ most people are able to store a maximum of 7 digits at a time;
- ▶ it is hard to keep intermediate results in mind, while performing further calculations;
- ▶ it is easier to operate with small numbers (1, 2, 3) than with greater (7, 8, 9).

Requirements for algorithms:

- ▶ complex operations are split into simpler ones;
- ▶ order of operations such that as few as possible intermediate results need to be stored;
- ▶ an approximative solution is earlier found and may serve as start for further iterations.

Outline



Addition & Subtraction

Multiplication

Squaring

Division

Squareroot

Logarithm

Sine & Co.

Verification

Guesstimation

Addition & Subtraction

the most basic



right to left

$$\begin{array}{r} 538 \\ + 327 \\ \hline 865 \end{array}$$

left to right

$$538 + 327 = 838 + 27 = 858 + 7 = 865$$

Multiplication



two equal digits, the other two complete to 10

$$ca_0 \cdot cb_0 \quad \text{with} \quad a_0 + b_0 = 10$$

or

$$a_1c \cdot b_1c \quad \text{with} \quad a_1 + b_1 = 10$$

$$a_1a_0 \cdot b_1b_0 = a_1 \cdot b_1 + c||a_0 \cdot b_0$$

$$74 \cdot 34 = 21 + 4||16 = 2516$$

$$47 \cdot 43 = 16 + 4||21 = 2021$$



Multiplication

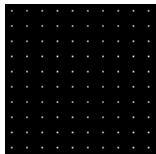
last digits 5



$$a_15 \cdot b_15 = a_1 \cdot b_1 + \frac{a_1 + b_1}{2} \parallel \begin{cases} 25 & \text{if } a_1 + b_1 = 2k \\ 75 & \text{else} \end{cases}$$

$$55 \cdot 35 = 15 + 4.0 \parallel 25 = 1925$$

$$65 \cdot 35 = 18 + 4.5 \parallel 75 = 2275$$



Multiplication

multiplication by 11



$$a_1 a_0 \cdot 11 = a_1 | a_1 + a_0 | a_0$$

$$32 \cdot 11 = 3|5|2 = 352$$

$$87 \cdot 11 = 8|15|7 = 957$$



Multiplication

numbers close by



$$(z + a) \cdot (z + b) = z \cdot (z + a + b) + a \cdot b$$

$$107 \cdot 108 = 100 \cdot 115 + 56 = 11556$$



special case middle number

$$(z + a) \cdot (z - a) = z^2 - a^2$$

$$61 \cdot 59 = 3600 - 1 = 3599$$

Multiplication

cross multiplication



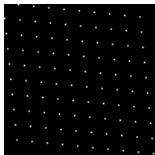
$$a_1 a_0 \cdot b_1 b_0 = a_1 b_1 \cdot 100 + (a_1 b_0 + a_0 b_1) \cdot 10 + a_0 \cdot b_0$$

$$23 \cdot 37 = 6 \cdot 100 + (14 + 9) \cdot 10 + 21 = 851$$

special case *mirror numbers*

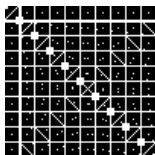
$$ab \cdot ba = a \cdot b \cdot 100 + (a^2 + b^2) \cdot 10 + a \cdot b$$

$$57 \cdot 75 = 35 \cdot 100 + (25 + 49) \cdot 10 + 35 = 4275$$



Multiplication

extensions



So far 25% covered

further coverage by different factorizations, e.g.

$$208 \cdot 53 = 104 \cdot 106 = 100 \cdot 110 + 24 = 11024 \quad (\text{close by}),$$

friendly intermediate products

$$21 \cdot 13 = 3 \cdot 7 \cdot 13 = 3 \cdot 91 = 273 \quad (\text{small factor last}),$$

$$89 \cdot 72 = 89 \cdot 9 \cdot 8 = 801 \cdot 8 = 6408 \quad (\text{zero in between})$$

or simple extensions

$$98 \cdot 9 = (100 - 2) \cdot 9 = 900 - 18 = 882.$$

Squaring

last digit 5



$$(a5)^2 = a \cdot (a + 1) || 25$$

$$45^2 = 4 \cdot 5 || 25 = 2025$$



Squaring

last digit 1 or 9



$$(a0 \pm 1)^2 = 10 \cdot a^2 \pm 2 \cdot a|1$$

$$41^2 = 160 + 8|1 = 1681$$

$$39^2 = 160 - 8|1 = 1521$$



Squaring

numbers near 50



$$(50 + a)^2 = 25 + a||a^2$$

$$53^2 = 25 + 3||09 = 2809$$

$$46^2 = 25 - 4||16 = 2116$$



Squaring

close by



$$(z + a)^2 = z \cdot (z + 2 \cdot a) + a^2$$

$$74^2 = 70 \cdot 78 + 16 = 5476$$



Squaring

extensions



simple cases $a \leq 11$

$$9^2 = 81$$

$$90^2 = 8100$$

all together 72%



Division

succession



$$\frac{a}{b} = n \cdot b + R$$

$$\frac{214}{7} = \mathbf{30} \cdot 7 + 4$$

$$\frac{4}{7} = \mathbf{0.5} \cdot 7 + 0.5$$

$$\frac{40}{7} \frac{1}{10}$$

$$\frac{0.5}{7} = \mathbf{0.07} \cdot 7 + 0.01$$

$$\frac{50}{7} \frac{1}{100}$$

...

$$\frac{214}{7} = \mathbf{30.57} \dots \quad (30.571428571)$$

Alternatively, learn basic fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ... by heart.

Squareroot

average with nearest integer squareroot



$$\sqrt{a} = \frac{1}{2}n + \frac{1}{2}\frac{a}{n}$$

with n^2 being the nearest square number to a , or a guess of it.

$$\sqrt{17} = \frac{4 + 4.25}{2} = 4.125 \quad (4.123105626)$$

$$\sqrt{30} = \frac{5 + 6}{2} = 5.5 \quad \text{another iteration}$$

$$= \frac{5.5 + 5.45}{2} = 5.475 \quad (5.477225575)$$

Logarithm



minimize required precomputed values by rearrangement

use approximation $\ln(1+x) \approx x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$

$$\begin{aligned}\log_{10} N &= \log_{10}((p_1 \cdot p_2 \cdot \dots)(1+x)) \\ &= \log_{10} e \cdot \ln((p_1 \cdot p_2 \cdot \dots)(1+x)) \\ &\approx \log_{10} e \cdot (\ln p_1 + \ln p_2 + \dots + x - \frac{1}{2}x^2 + \dots)\end{aligned}$$

$$\begin{aligned}\log_{10} 1211 &= \log_{10}(1200 \cdot (1 + 11/1200)) \\ &= \log_{10}(100 \cdot 12 \cdot (1 + 11/1200)) \\ &= 2 + \log_{10}(12 \cdot (1 + 11/1200)) \\ &= 2 + \log_{10}(2^2 \cdot 3 \cdot (1 + 11/1200)) \\ &= 2 + \log_{10} e \cdot \ln(2^2 \cdot 3 \cdot (1 + 11/1200)) \\ &\approx 2 + \log_{10} e \cdot (2 \ln 2 + \ln 3 + 11/1200) \\ &\approx 2 + \frac{1}{2.3} \cdot (2 \cdot 0.7 + 1.1 + 0.01) \\ &\approx 3.1 \quad (3.083144143)\end{aligned}$$

Interest Rate

doubling time of investments



Years for doubling at an annual interest rate of $P\% < 25\%$

$$y = 70/P \quad (y = \log_{1+p} 2) \quad \text{with} \quad (p = P/100)$$

$$P = 5\% \quad \rightsquigarrow \quad y = \frac{70}{5} = 14 \quad (14.2066990)$$

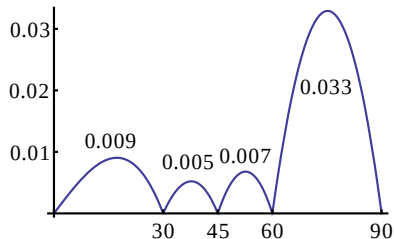
Sine & Co. I



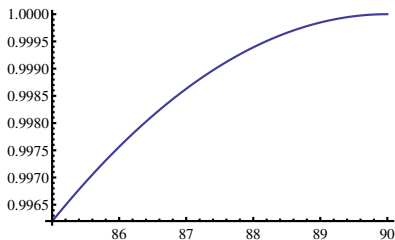
linear interpolation and visual correction

linear interpolation

x	0°	30°	45°	60°	90°
$\sin(x)$	$\frac{\sqrt{0}}{2} = 0$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{2}}{2} \approx \frac{707}{1000}$	$\frac{\sqrt{3}}{2} \approx \frac{866}{1000}$	$\frac{\sqrt{4}}{2} = 1$



sine minus linear interpolation



sine close to maximum

$$\sin(57^\circ) = 0.834 + 0.004 = 0.838 \quad (0.838671)$$

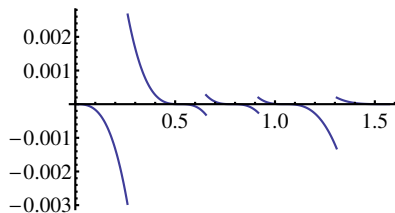
Sine & Co. II

alternatives

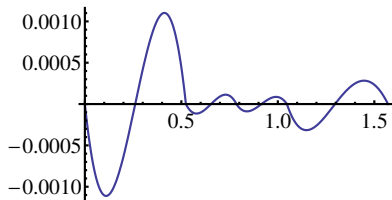


Taylor expansion around $x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$

$$\sin(x + \Delta x) \approx \sin x + \Delta x \cos x - \frac{(\Delta x)^2}{2} \sin x$$



taylor expansion error



polynomial fit error

piecewise polynomial fit reduces error further ($\epsilon \approx 10^{-3}$) with same computational effort but more precomputed values (coefficients)

$$\sin(x + \Delta x) \approx a_2(\Delta x)^2 + a_1\Delta x + \sin x$$

Verification



casting out 9s for addition and multiplication (division)

$$\text{mod}(a, 9) + \text{mod}(b, 9) = \text{mod}(a + b, 9)$$

$$\text{mod}(a, 9) \cdot \text{mod}(b, 9) = \text{mod}(a \cdot b, 9)$$

$$\text{mod}(a_n a_{n-1} a_{n-2} \dots a_2 a_1 a_0, 9) = \sum_{i=0}^n a_i \quad \text{sum of digits}$$

$$21 + 31 \equiv 52 \pmod{9}$$

$$(2 + 1) + (3 + 1) \equiv (5 + 2) \pmod{9}$$

$$7 \equiv 7 \pmod{9}$$

$$21 \cdot 31 \equiv 651 \pmod{9}$$

$$(2 + 1) \cdot (3 + 1) \equiv 6 + 5 + 1 \pmod{9}$$

$$12 \equiv 12 \pmod{9}$$

$$3 \equiv 3 \pmod{9}$$

Verification

casting out 11s, 99s or 101s



$$\begin{array}{rcl} 21 \cdot 31 & \equiv & 651 \quad \text{mod } 11 \\ (-2 + 1) \cdot (-3 + 1) & \equiv & 6 - 5 + 1 \quad \text{mod } 11 \\ 2 & \equiv & 2 \quad \text{mod } 11 \end{array}$$

$$\begin{array}{rcl} 229 \cdot 721 & \equiv & 165109 \quad \text{mod } 99 \\ (02 + 29) \cdot (07 + 21) & \equiv & 16 + 51 + 09 \quad \text{mod } 99 \\ 868 & \equiv & 76 \quad \text{mod } 99 \\ 08 + 68 & \equiv & 76 \quad \text{mod } 99 \end{array}$$

$$\begin{array}{rcl} 229 \cdot 721 & \equiv & 165109 \quad \text{mod } 101 \\ (-02 + 29) \cdot (-07 + 21) & \equiv & 16 - 51 + 09 \quad \text{mod } 101 \\ 378 & \equiv & -26 \quad \text{mod } 101 \\ (-03 + 78) & \equiv & 75 \quad \text{mod } 101 \end{array}$$

Guesstimation



if moderately accurate solutions are sufficient

$$8\,367 + 5\,819 \approx 8\,000 + 6\,000 = 14\,000 \quad (14\,186)$$

$$1.39 + 0.87 + 2.46 \approx 1.50 + 1.00 + 2.50 = 5.00 \quad (4.72)$$

$$\frac{5\,000\,000}{365} = \frac{500}{365} 10\,000 \approx 1.4 \cdot 10\,000 \quad (13\,698.63)$$

$$73 \cdot 65 \approx \begin{cases} 70 \cdot 68 = 4760 \\ 78 \cdot 60 = 4680 \end{cases} \quad (4745)$$

The fun starts here



- ▶ Summary

- ▶ mental math - elementary operations [1]

In order to extend the small 1×1 to the big 1×1 , you need to learn special cases, how to recognize and rearrange them quickly.

- ▶ Outlook

- ▶ memory training [2]
 - ▶ mental math - advanced calculus [3, 4]

shell script



```
#!/bin/bash
```

```
now=$(date)
```

```
exercises=$(~/projects/krt/build/krt_F)
```

```
thunderbird -compose
```

```
"from=sender@xyz.com,
```

```
to=receiver@xyz.com,
```

```
subject='$now',body='$exercises'"
```

References



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